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The Total Variation of the Tensor Product Bernstein–Bézier Operator

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IN MEMORY OF I. J. SCHOENBERG (1903–1990)

An inequality bounding the total variation of the tensor product Bernstein–Bézier polynomial is given.

INTRODUCTION

We prove a variation diminishing property of the bivariate tensor-product Bernstein–Bézier (BB) operator. In particular, we show that the total (first-order) variation of a "net" of coefficients of a bivariate tensor-product BB polynomial bounds the total variation of the polynomial from above. This result supplements a series of results that have appeared in [ChDa84, G87a, b, CaSh91] and that relate the convexity and variation of bivariate BB polynomials to their corresponding nets of coefficients: [ChDa84] showed that convexity of bivariate BB polynomials (over a triangle) is implied by the convexity of the corresponding BB net; [G87a, b] also consider the same BB operator and prove total variation diminuition and variation diminuition with respect to arbitrary seminorms; [CaSh91] extends most of the above results to bivariate, tensor-product, BB operators, but leaves open the question of total variation diminuition. This question is settled in the current note. Our proof relies mainly on analyzing the de Casteljau subdivision process that generates the tensor-product polynomial from the net of its BB coefficients.

PRELIMINARIES

For a C^1 function f on the unit square R, the *total variation* $V(f; R)$ is given by

$$V(f; R) := \iint_R \sqrt{f_x^2 + f_y^2}\, dx\, dy.$$

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For positive integers n, m the tensor product BB polynomial for f is

$$Bf(x, y) := \sum_{i=0}^{n} \sum_{j=0}^{m} f_{ij}\, c_{ni}(x)\, c_{mj}(y),$$

where

$$c_{ni}(u) := \binom{n}{i} u^i (1-u)^{n-i}$$

and

$$f_{ij} := f\left(\frac{i}{n}, \frac{j}{m}\right).$$

We introduce the function $\hat{f} := \hat{f}_{nm}$ called the *BB net* determined by the values f_{ij}. We define $\hat{f}$ on each of mn rectangles, R_{ij}, $i \in \{0, 1, ..., n-1\}$, $j \in \{0, 1, ..., m-1\}$, that are obtained by partitioning R using the grid defined by the lines $x = i/n$, $i \in \{0, 1, ..., n\}$, and $y = j/m$, $j \in \{0, 1, ..., m\}$, as follows:

$$\begin{aligned}\hat{f}(x, y)|_{R_{ij}} := {} & f_{ij}(i+1-nx)(j+1-my) + f_{i+1,j}(nx-i)(j+1-my) \\ & + f_{i,j+1}(my-j)(i+1-nx) + f_{i+1,j+1}(nx-i)(my-j).\end{aligned}$$

Clearly $\hat{f}$ is continuous, interpolates f at the grid points, and is bilinear on each rectangle R_{ij}. A direct calculation shows that

$$\begin{aligned}V(\hat{f}; R) = \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \int_0^1 \int_0^1 \bigg(& \frac{1}{m^2} (\delta^1 f_{ij}(1-u) + \delta^1 f_{i,j+1} u)^2 \\ & + \frac{1}{n^2} (\delta^2 f_{ij}(1-v) + \delta^2 f_{i+1,j} v)^2 \bigg)^{1/2} du\, dv, \end{aligned} \tag{1.1}$$

where

$$\delta^1 f_{ij} = f_{i+1,j} - f_{ij}, \qquad i \in \{0, ..., n-1\}, \qquad j \in \{0, ..., m\}$$

and

$$\delta^2 f_{ij} = f_{i,j+1} - f_{ij}, \qquad i \in \{0, ..., n\}, \qquad j \in \{0, ..., m-1\}.$$

Note. In the following, we assume that the positive integers n and m are fixed, and hence, in general, omit them from subscripts.

Total Variation Diminuition

Our goal here is to prove the following.

Theorem. *For any $f \in C^1(R)$,*

$$V(Bf; R) \leqslant V(\hat{f}; R).$$

Note that this inequality clearly is an equality whenever Bf is bilinear. The proof will rely on using this as a ground case. The proof also depends on de Casteljau subdivision techniques, developed first for studying the variation of BB operator on triangles by Goodman [G87a, b] and further extended to the tensor-product case by Cavaretta and Sharma [CaSh91]. As in this latter paper, we subdivide the unit square into quarters by the lines $x = \frac{1}{2}$ and $y = \frac{1}{2}$, thus obtaining four squares $R^j, j = 1, 2, 3, 4$. We denote the BB basis functions for R^r by β^r_{ij} (these are analogous to $c_{ni}(x)\, c_{mj}(y)$ defined earlier). For example,

$$\beta^1_{ij}(x, y) = 2^{n+m} \binom{n}{i} \binom{m}{j} x^i y^j \left(\frac{1}{2} - x\right)^{n-i} \left(\frac{1}{2} - y\right)^{m-j}.$$

For any array b_{ij} we use the averaging operators

$$(A_1 b)_{ij} = \tfrac{1}{2}(b_{i+1,j} + b_{i-1,j})$$
$$(A_2 b)_{ij} = \tfrac{1}{2}(b_{i,j+1} + b_{i,j-1}).$$

Then the algebraic content of the de Casteljau algorithm is embodied in the following.

Lemma. *Let $\{b_{ij}\}_{i,j=0}^{2n,2m}$ be any array satisfying $b_{2i,2j} = f_{ij}$ and suppose*

$$Bf(x, y)\,|_{R^r} = \sum_{i=0}^{n} \sum_{j=0}^{m} a^r_{ij}\, \beta^r_{ij}(x, y), \qquad r = 1, 2, 3, 4. \tag{1.2}$$

Then for $i = 0, ..., n$ and $j = 0, ..., m$,

$$\begin{aligned} a^1_{ij} &= (A^i_1 A^j_2 b)_{ij} & a^2_{ij} &= (A^{k-i}_1 A^j_2 b)_{n+i,j} \\ a^3_{ij} &= (A^{n-i}_1 A^{m-j}_2 b)_{n+i,m+j} & a^4_{ij} &= (A^i_1 A^{m-j}_2 b)_{i,m+j}. \end{aligned} \tag{1.3}$$

For the proof of these formulas, simply compute the indicated iterated averages in terms of binomial coefficients and then compare the result to the Casteljau algorithm. For details, see [CaSh 91]. Because of the identity

$$\int_0^1 \int_0^1 Bf(x, y)\, dx\, dy = \sum_{r=1}^{4} \iint_{R^r} Bf(x, y)\,|_{R^r}\, dx\, dy$$

we obtain the following.

COROLLARY. *For any array* $\{b_{ij}\}_{i=0, j=0}^{2n, 2m}$ *the following algebraic identity holds*:

$$\sum_{i=0}^{n} \sum_{j=0}^{m} \{(A_1^i A_2^j b)_{ij} + (A_1^{n-i} A_2^j b)_{n+i,j} + (A_1^{n-i} A_2^{m-j} b)_{n+i,m+j} + (A_1^i A_2^{m-j} b)_{i,m+j}\} = 4 \sum_{i=0}^{n} \sum_{j=0}^{m} b_{2i,2j}. \tag{1.4}$$

Formulas (1.2) and (1.3) naturally define a subdivision operation on the net $\hat{f}$. Each of the four sequences $\{a_{ij}^r\}_{i,j=0}^{n,m}$, determines a net $\hat{g}^r = \hat{g}_{nm}^r$ on R^r, $r = 1, 2, 3, 4$. The following proposition gives an important variation-diminishing property of the subdivision algorithm.

PROPOSITION. $\sum_{r=1}^4 V(\hat{g}^r; R^r) \leqslant V(\hat{f}; R)$.

Proof. Using (1.1) we find that for $r = 1, \ldots, 4$

$$V(\hat{g}^r; R^r) = \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} z_{ij}^r, \tag{1.5}$$

where

$$z_{ij}^r = \frac{1}{2} \int_0^1 \int_0^1 \left\{ \frac{1}{m^2} (\delta_1 a_{ij}^r (1-u) + \delta_1 a_{i,j+1}^r u)^2 + \frac{1}{n^2} (\delta^2 a_{ij}^r (1-v) + \delta^2 a_{i+1,j}^r v)^2 \right\} du\, dv. \tag{1.6}$$

Using (1.3), and the array $\{b_{ij}\}_{i,j=0}^{2n,2m}$ we find for $r = 1$ that

$$\delta_1 a_{ij}^1 = \frac{A_1^i A_2^j}{2} (b_{i+2,j} - b_{ij})$$

and upon further calculation we obtain

$$\delta_1 a_{ij}^1 (1-u) + \delta^1 a_{i,j+1}^1 u = \frac{A_1^i A_2^j}{2} \left\{ (b_{i+2,j} - b_{ij}) \left(1 - \frac{u}{2}\right) + (b_{i+2,j+2} - b_{i,j+2}) \frac{u}{2} \right\}.$$

Similarly

$$\delta^2 a_{ij}^1 (1-v) + \delta^2 a_{i+1,j}^1 v = \frac{A_1^i A_2^j}{2} \left\{ (b_{i,j+2} - b_{ij}) \left(1 - \frac{v}{2}\right) + (b_{i+2,j+2} - b_{i+2,j}) \frac{v}{2} \right\}.$$

Using these formulas in (1.6) we obtain from the triangle inequality and an appropriate change of variables that

$$z_{ij}^1 \leqslant A_1^i A_2^j w_{ij},$$

where

$$w_{ij} := \int_0^{1/2} \int_0^{1/2} \left[\left\{ \frac{1}{m} (b_{i+2,j} - b_{ij})(1-u) + (b_{i+2,j+2} - b_{i,j+2}) u \right\}^2 \right.$$
$$\left. + \left\{ \frac{1}{n} (b_{i,j+2} - b_{ij})(1-v) + (b_{i+2,j+2} - b_{i+2,j}) v \right\}^2 \right]^{1/2} du\, dv.$$

Similarly one finds that

$$z_{ij}^2 \leqslant A_1^{n-i-1} A_2^j w_{n+i-1,j}$$
$$z_{ij}^3 \leqslant A_1^{n-i-1} A_2^{m-j-1} w_{n+i-1,m+j-1}$$
$$z_{ij}^4 \leqslant A_1^i A_2^{m-j-1} w_{i,m+j-1}.$$

Using these inequalities in (1.5) and then applying the corollary with $n-1$, $m-1$ in place of n and m we get

$$\sum_{r=1}^{4} V(\hat{g}^r; R^r) \leqslant 4 \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} w_{2i,2j} = 4V(\hat{f}; Q^1),$$

where $Q^1 \subset R$ is defined by the equations

$$Q^1 \cap \rho_{ij} = \left(\frac{i}{n}, \frac{i}{n} + \frac{1}{2n} \right) \times \left(\frac{j}{m}, \frac{j}{m} + \frac{1}{2m} \right),$$
$$i \in \{0, ..., n-1\}, \qquad j \in \{0, ..., m-1\}.$$

By applying the same argument to $f(x, 1-y)$, $f(1-x, y)$, and $f(1-x, 1-y)$, we obtain the inequalities

$$\sum_{r=1}^{4} V(\hat{g}^r; R^r) \leqslant 4V(\hat{f}; Q^s), \qquad s \in \{1, 2, 3, 4\},$$

where Q^s, $s \in \{2, 3, 4\}$, are the images of Q^1 under the various similarity transformations obtained from $x \to 1-x$, $y \to 1-y$. Averaging these four inequalities proves the proposition. ∎

We can now prove the main theorem.

Proof of Theorem. First note that since both $V(\hat{f})$ and $V(Bf)$ are norms on the finite dimensional space of polynomials of degree (m, n) modulo the constants, we have for some $c > 0$

$$V(\hat{f}; R) \leqslant cV(Bf; R)$$

for all continuous f. Moreover, the inequality is shift and scale invariant and therefore valid on any square σ.

Let $l \geqslant 1$ be any integer and partition R into l^2 subsquares. If σ denotes any one of these subsquares, then

$$Bf|_\sigma = q + r,$$

where q is bilinear and r, $\partial r/\partial x$, $\partial r/\partial y$, and $\partial^2 r/\partial x\, \partial y$ all vanish at some point of σ. Therefore

$$V(Bf; \sigma) = V(q; \sigma) + o\left(\frac{1}{l^2}\right). \tag{1.7}$$

On σ, choose g so that

$$Bf|_\sigma = B(g; \sigma).$$

Then since q is bilinear, it follows that $\hat{q} = q$ on σ, and hence

$$V(q; \sigma) \leqslant V(\hat{g}; \sigma) + V((\hat{q} - \hat{g}); \sigma) \leqslant V(\hat{g}; \sigma) + cV(B(q-g); \sigma).$$

Since $B(q-g) = q - Bf|_\sigma = -r$ it follows that

$$V(q; \sigma) \leqslant V(\hat{g}; \sigma) + o\left(\frac{1}{l^2}\right).$$

Using this inequality in (1.7) and summing over all σ, we obtain

$$V(Bf; R) \leqslant \sum_\sigma V(\hat{g}; \sigma) + o(1).$$

If we choose $l = 2^s$, then the proposition can be applied s times to yield

$$V(Bf: R) \leqslant V(\hat{f}; R) + o(1)$$

from which the theorem follows by letting s tend to infinity. ∎

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